On the mass of a Kerr-anti-de Sitter spacetime in D dimensions

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Abstract

We show how to compute the mass of a Kerr-anti-de Sitter spacetime with respect to the anti-de Sitter background in any dimension, using a superpotential which has been derived from standard Nœther identities. The calculation takes no account of the source of the curvature and confirms results obtained for black holes via the first law of thermodynamics.

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In an interesting paper Gibbons, Lu, Page and Pope [1] generalized the Kerr-(anti)-de Sitter metrics to all dimensions $D > 5$. Furthermore Gibbons, Perry and Pope [2] calculated the angular momenta $J_{(D)i}$ of these rotating spacetimes using Komar's integrals. They also calculated the mass $E_{(D)}$ of rotating *black holes* in anti-de Sitter backgrounds using the first law of thermodynamics.

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Let us emphazise that $E_{(D)}$ and $J_{(D)i}$ are classical concepts on which thermodynamics is built rather than the other way round. A Kerr-anti-de Sitter spacetime with the same mass might be produced by a rotating "star" rather than a black hole. Such a body has neither quantum radiation nor a Bekenstein-Hawking entropy. One should surely be able to calculate the mass in that case because the spacetime admits a timelike Killing vector.

Here we show how to calculate the mass and the angular momenta of Kerr-anti-de Sitter metrics using the covariant KBL superpotential [3] (see [4] about the uniqueness and reliability of that superpotential).

We choose to write the D-dimensional Kerr-anti-de Sitter metrics in Kerr-Schild coordinates. In odd dimensions $D = 2n + 1$ for instance, the line element reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = d\bar{s}^2 + \frac{2m}{U} (h_\mu dx^\mu)^2 \qquad \text{with} \quad \mu = \{0, 1, \mu_i, \phi_i\} \quad (i = 1, ..., n) \tag{1}$$

where $d\bar{s}^2$ is the metric of the anti-de Sitter background and where the coordinates μ_i are subject to the constraint $\sum_{i=1}^{i=n} \mu_i^2 = 1$. The mass parameter m is an integration constant, h^μ is some nul vector (denoted k^μ in [1-2]) depending on n rotation parameters a_i and U is a scalar function. Explicit forms of those functions can be found in Section 2 of ref [1]. These coordinates are the most convenient for our calculations. The background admits a timelike Killing vector ξ and n plane rotation Killing vectors η_i associated with one parameter (say τ) displacements. ξ is normalized in such a way that $\xi^\mu \delta\tau$ is an infinitesimal time translation in a locally free falling *non rotating* inertial frame in the background. Thus for the background metric in its conventional static form when $D = 2n + 1$ for instance (equations (1.11) and (1.12) of Ref [1])

$$d\bar{s}^2 = -\left(1 + \frac{y^2}{l^2}\right) dt^2 + \frac{dy^2}{1 + y^2/l^2} + y^2 d\sigma^2_{(D-2)}$$

$$\text{where} \qquad d\sigma^2_{(D-2)} = \sum_{i=1}^{i=n} (d\tilde{\mu}_i^2 + \tilde{\mu}_i^2 d\phi_i^2) \quad \text{with} \quad \sum_{i=1}^{i=n} \tilde{\mu}_i^2 = 1, \tag{2}$$

the components of the timelike Killing vector ξ corresponding to time translations are $(1, 0, 0, 0, ...)$. In Kerr-Schild spheroidal coordinates ξ has the same components. On the other hand, in Boyer-Linquist coordinates used in Section 3 of ref [1], the components are $(\xi^0 = 1, \xi^1 = 0, \xi^{\mu_i} = 0, \xi^{\phi_i} = a_i/l^2)$.

We now turn to the calculation of $E_{(D)}$ using the KBL superpotential $\hat{J}^{[\mu\nu]}$ (where a hat means multiplication by $\sqrt{-g}$ and brackets denote antisymetrisation) which is defined as (see [5] for a recent review and references) :

$$\hat{J}^{[\mu\nu]} = -\frac{1}{8\pi} \left(D^{[\mu} \hat{\xi}^{\nu]} - \overline{D^{[\mu} \hat{\xi}^{\nu]}} + \hat{\xi}^{[\mu} k^{\nu]} \right) \tag{3}$$

with a "Newton constant" $G_D \equiv 1$. The first term in (3) is the well known Komar superpotential density of the foreground with metric $\mathbf{g}$. The second term substracts the

Komar superpotential of the background with metric $\bar{\mathbf{g}}$. The vector k^ν in the third term is :

$$k^\nu = g^{\nu\rho}(\Gamma^\sigma_{\rho\sigma} - \overline{\Gamma^\sigma_{\rho\sigma}}) - g^{\rho\sigma}(\Gamma^\nu_{\rho\sigma} - \overline{\Gamma^\nu_{\rho\sigma}})\,. \tag{4}$$

The mass $E_{(D)}$ is given by the integral of $\hat{J}^{[\mu\nu]}$ on a $(D-2)$ surface S, of "radius" $x^1 = r \to \infty$ in a $(D-1)$ hypersurface $x^0 = t = const.$ in Kerr-Schild coordinates :

$$E_{(D)} = \int_S d^{D-2}x\, \hat{J}^{[01]}\,. \tag{5}$$

We repeat that the integrand being covariant, the integral is a scalar which may be calculated in any convenient coordinates.

Angular momenta $J_{(D)i}$ are obtained by replacing ξ by η_i in the KBL superpotential $\hat{J}^{[01]}$. Since $\eta_i^0 = \eta_i^1 = 0$ it reduces to the Komar superpotentials and :

$$J_{(D)i} = \int_S d^{D-2}x \hat{J}_i^{[01]} \quad \text{with} \quad \hat{J}_i^{[01]} = -\frac{1}{8\pi}[\hat{D}^{[0}\eta_i^{1]} - \overline{\hat{D}^{[0}\eta_i^{1]}}]\,. \tag{6}$$

The calculation of $\hat{J}^{[01]}$ for the mass is a mechanical procedure, which is straightforward in Kerr-Schild coordinates.* Using the asymptotic form of the metric given in [1] at leading order in r we arrive at :

$$\hat{J}^{[01]} = \frac{m}{8\pi\Xi}\left[(D-1)W - 1\right]\sqrt{g_{(D-2)}} + \mathcal{O}\left(\frac{m^2}{r}\right)\,. \tag{7}$$

$g_{(D-2)}$ is the determinant of the metric $d\sigma^2_{(D-2)}$ in (2) with tildes dropped and

$$\Xi \equiv \prod_{i=1}^{i=n} \Xi_i \quad , \quad W = \sum_{i=1}^{i=n} \frac{\mu_i^2}{\Xi_i} \quad , \quad \Xi_i = 1 - \frac{a_i^2}{l^2}$$

with $a_n = 0$ in even dimensions $D = 2n$.

As for the calculation of $\hat{J}_i^{[01]}$ for the angular momenta, it readily yields (the background term $\overline{\hat{D}^{[0}\eta_i^{1]}}$ does not contribute) :

$$\hat{J}_i^{[01]} = \frac{m}{8\pi\Xi}\frac{a_i\mu_i^2}{\Xi_i}(D-1)\sqrt{g_{(D-2)}} + \mathcal{O}\left(\frac{m^2}{r}\right)\,. \tag{8}$$

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* The useful properties are that the metric coefficients do not depend on x^0 and ϕ_i; the $g_{\mu_i\mu_j}$ coefficients do not depend on m; the other off-diagonal components are linear in m and hence do not enter the calculation. The fact that the vector h^μ is nul further simplifies the calculation since $\sqrt{-g} = \sqrt{-\bar{g}}$.

The explicit integration of (7-8) with all angles $\phi_i \in [0, 2\pi]$ (ϕ_n being absent in $D = 2n$ dimensions), with μ_n eliminated by means of the constraint and with all $\mu_i \in [0, 1]$ apart from $\mu_{(n-1)} \in [-1, +1]$ in $D = 2n$ dimensions then yields :

$$E_{(4)} = \frac{m}{\Xi^2} \quad , \quad E_{(5)} = \frac{\pi m}{4(\Xi_1\Xi_2)^2}(2\Xi_1 + 2\Xi_2 - \Xi_1\Xi_2) \quad , \quad E_{(6)} = \frac{2\pi m}{3\,\Xi_1\Xi_2}\left(\frac{1}{\Xi_1} + \frac{1}{\Xi_2}\right),$$

$$J_{(4)} = \frac{ma}{\Xi^2} \quad , \quad J_{(5)i} = \frac{\pi m a_i}{2\,\Xi\,\Xi_i} \quad , \quad J_{(6)i} = \frac{2\pi m a_i}{3\,\Xi\,\Xi_i}\,, \tag{9}$$

where $\Xi = 1 - a^2/l^2$ for $D = 4$ and where $\Xi = \Xi_1\Xi_2$ and $i = 1$ or 2 in $D = 5$ and $D = 6$. It is not hard to calculate the expressions for $E_{(D)}$ and $J_{(D)i}$ for any even D. For odd dimensions some care must be exercised. The results agree, for all even D and, explicitely, all odd D up to $D = 11$, with the general formulae given in [2], to wit

$$E_{2n} = \frac{m\mathcal{V}_{2n-2}}{4\pi\Xi}\sum_{i=1}^{i=n-1}\frac{1}{\Xi_i} \quad , \quad E_{2n+1} = \frac{m\mathcal{V}_{2n-1}}{4\pi\Xi}\left(\sum_{i=1}^{1=n}\frac{1}{\Xi_i} - \frac{1}{2}\right) \quad , \quad J_{(D)i} = \frac{m\mathcal{V}_{D-2}}{4\pi\Xi}\frac{a_i}{\Xi_i}$$

where $\mathcal{V}_{D-2}$ is the volume of the $(D-2)$-sphere :

$$\mathcal{V}_{D-2} = \frac{2\,\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}\,.$$

These results confirm those given in [2]. However they were obtained using standard ideas about global Nœther conservation laws associated with translational and rotational invariance of a background spacetime. They depend on the asymptotic form of the metric only and hence take no account of the source of the curvature. In [2] the mass was obtained by using the first law of black hole thermodynamics given that the entropy is identified with 1/4 of the black hole area and the temperature with its surface gravity. Here we know the mass in the first place and assuming the same temperature as in [2] we may compute the entropy of a black hole if there is one.

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